Solutions

Exam 3 Review: Sections 3.5-3.7 and 7.1-7.5

Section 3.5 Find the general solution to the differential equation

$$y'' + 2y' + 2y = 3x^2 - 1.$$

$r^2 + 2r + 2 = 0$

$(r+1)^2 + 1 = 0$

$r = -1 \pm i$

So $y_c = e^{-x}(c_1 \cos x + c_2 \sin x)$.

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$\Rightarrow$ $3x^2 - 1 = 2A + 2(2Ax + B) + 2(Ax^2 + Bx + C)$

$= 2Ax^2 + (4A + 2B)x + 2A + 2B + 2C$

So $3 = 2A \Rightarrow A = 3/2$

$0 = 4A + 2B = 6 + 2B \Rightarrow B = -3$

$-1 = 2A + 2B + 2C = 3 - 6 + 2C \Rightarrow C = 1$

So $y = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{3}{2}x^2 - 3x + 1$.

Section 3.5 Find the general form of the particular solution for the differential equation

$$y^{(4)} - 5y'' + 4y = e^x - xe^{2x}.$$

(Do not solve for the undetermined coefficients.)

$r^4 - 5r^2 + 4 = 0$

$(r^2 - 4)(r^2 - 1) = 0$

$(r-2)(r+2)(r-1)(r+1) = 0$

So $y_c = c_1 e^{-2x} + c_2 e^{2x} + c_3 e^{-x} + c_4 e^x$

First guess

$y_p = Ae^x + Be^{2x} + Cxe^{2x}$

Adjusted and final form

$y_p = Axe^x + Bxe^{2x} + Cx^2e^{2x}$.

Section 3.6 Consider an undamped mass-and-spring system with mass $m = 1$, spring constant $k = 100$ and forced oscillations given by the function $F(t) = 225\cos 5t + 300\sin 5t$. Assume further that $x(0) = 375$ and $x'(0) = 0$. Find the solution function $x(t)$.

$$mx'' + cx' + kx = F(t) \Rightarrow x'' + 100x = 225\cos 5t + 300\sin 5t$$

$$x_c = c_1\cos 10t + c_2\sin 10t$$

$$x_p = A\cos 5t + B\sin 5t$$

$$x_p'' = -25A\cos 5t - 25B\sin 5t$$

So $225\cos 5t + 300\sin 5t = -25A\cos 5t - 25B\sin 5t + 100(A\cos 5t + B\sin 5t)$

$$= 75A\cos 5t + 75B\sin 5t.$$

Thus $A = 3$ and $B = 4$.

So $x = c_1\cos 10t + c_2\sin 10t + 3\cos 5t + 4\sin 5t.$

$$x(0) = 375 = c_1 + 3 \Rightarrow c_1 = 372$$

$$x' = -3720\sin 10t + 10c_2\cos 10t - 15\sin 5t + 20\cos 5t$$

$$x'(0) = 0 = 10c_2 + 20 \Rightarrow c_2 = -2.$$

Therefore

$$x = 372\cos 10t - 2\sin 10t + 3\cos 5t + 4\sin 5t.$$

Sections 3.7 Consider an RLC circuit with $R = 40$ ohms, $L = 10$ henries, $C = 0.02$ farads and $E(t) = 50 \sin 2t$ volts at time t. This information gives the differential equation

$$10I'' + 40I' + 50I = 100 \cos 2t$$

for the current $I(t)$ (in amperes). Find the general complementary solution and the particular solution for this circuit.

$$I'' + 4I' + 5I = 10\cos 2t$$

$$r^2 + 4r + 5 = 0$$

$$(r+4)(r+1) = 0$$

So $I_c = c_1 e^{-4t} + c_2 e^{-t}$.

$$I_p = A\cos 2t + B\sin 2t$$

$$I_p' = -2A\sin 2t + 2B\cos 2t$$

$$I_p'' = -4A\cos 2t - 4B\sin 2t$$

$$10\cos 2t = -4A\cos 2t - 4B\sin 2t + 4(-2A\sin 2t + 2B\cos 2t) + 5(A\cos 2t + B\sin 2t)$$

$$= (A+8B)\cos 2t + (B-8A)\sin 2t.$$

$10 = A + 8B = 65A \Rightarrow A = \frac{10}{65} = \frac{2}{13}$ and $B = \frac{16}{13}$.

$0 = B - 8A \Rightarrow B = 8A$

Thus $I = c_1 e^{-4t} + c_2 e^{-t} + \frac{2}{13}\cos 2t + \frac{16}{13}\sin 2t.$

Section 7.1 State the Laplace transform of the following functions.

1. $f(t) = 1$
2. $g(t) = e^{-3t}$
3. $h(t) = t^{11}$
4. $i(t) = \sin 4t$
5. $j(t) = \cos 5t$

$$\mathcal{L}\{f(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{g(t)\} = \frac{1}{s+3}$$

$$\mathcal{L}\{h(t)\} = \frac{11!}{s^{12}}$$

$$\mathcal{L}\{i(t)\} = \frac{4}{s^2+16}$$

$$\mathcal{L}\{j(t)\} = \frac{s}{s^2+25}.$$

Section 7.1 Find the inverse Laplace transform for the function $F(s) = \frac{3s+1}{s^2+4}$.

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= 3\cos 2t + \frac{1}{2}\sin 2t.$$

Sections 7.2 Use Laplace transforms to solve the initial value problem

$$x'' + 8x' + 25x = 0; \quad x(0) = 2, x'(0) = 3.$$

$\mathcal{L}\{0\} = \mathcal{L}\{x'' + 8x' + 25x\}$

$0 = (s^2X - 2s - 3) + 8(sX - 2) + 25X$

$X(s^2 + 8s + 25) = 2s + 19$

$X(s) = \frac{2s+19}{s^2+8s+25} = \frac{2s+19}{(s+4)^2+9} = \frac{2(s+4)}{(s+4)^2+9} + \frac{11}{(s+4)^2+9}$

So $x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{-4t}\left(2\cos 3t + \frac{11}{3}\sin 3t\right)$.

Section 7.2 Find the inverse Laplace transform for the function $F(s) = \frac{1}{s^2(s^2+1)}$.

$\mathcal{L}^{-1}\{F(s)\} = \int_0^t \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}}(\tau)\, d\tau$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}(\tau)\, d\tau$

$= \int_0^t \sin\tau \, d\tau = -\cos\tau \Big|_0^t = 1 - \cos t.$

So $\mathcal{L}^{-1}\{F(s)\} = \int_0^t 1 - \cos\tau \, d\tau$

$= \tau - \sin\tau \Big|_0^t = t - \sin t.$

Section 7.3 Find the inverse Laplace transform for the function $F(s) = \frac{5s-4}{s^3-s^2-2s}$.

$$F(s) = \frac{5s-4}{s^3-s^2-2s} = \frac{5s-4}{s(s-2)(s+1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+1}.$$

So $5s-4 = A(s-2)(s+1) + Bs(s+1) + Cs(s-2)$

$= (A+B+C)s^2 + (-A+B-2C)s - 2A$

and hence $0 = A+B+C = 2+B+C$ $\qquad 0 = 9+3C \Rightarrow C = -3$

$5 = -A+B-2C = -2+B-2C \Rightarrow B = 7+2C \Rightarrow B = 1$

$-4 = -2A \Rightarrow A = 2$

So $F(s) = \frac{2}{s} + \frac{1}{s-2} - \frac{3}{s+1}$.

Therefore $\mathcal{L}^{-1}\{F(s)\} = 2 + e^{2t} - 3e^{-t}$.

Section 7.3 Find the Laplace transform for the function $f(t) = t^4 e^{\pi t}$.

Let $F(s) = \mathcal{L}\{t^4\} = \frac{4!}{s^5}$.

Then $\mathcal{L}\{t^4 e^{\pi t}\} = F(s-\pi) = \frac{4!}{(s-\pi)^5}$.

Section 7.4 Use the fact that $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ to find

$$\mathcal{L}^{-1}\{\frac{98}{(s-2)(s-3)}\}.$$

$$\frac{98}{(s-2)(s-3)} = 98 \cdot \frac{1}{s-2} \cdot \frac{1}{s-3} = 98 \cdot \mathcal{L}\{e^{2t}\} \cdot \mathcal{L}\{e^{3t}\}.$$

So $\mathcal{L}^{-1}\left\{\frac{98}{(s-2)(s-3)}\right\} = 98(e^{2t}) * (e^{3t}) = 98\int_0^t e^{2(t-x)} e^{3x}\,dx = 98e^{2t}\int_0^t e^x\,dx$

$$= 98e^{2t} \cdot e^x \Big|_0^t = 98e^{2t}(e^t - 1) = 98(e^{3t} - e^{2t}).$$

Section 7.4 Use Laplace transforms to find a non-trivial solution to the differential equation

$$tx'' + (3t-1)x' + 3x = 0; \quad x(0) = 0.$$

$$\mathcal{L}\{0\} = \mathcal{L}\{tx'' + (3t-1)x' + 3x\}$$

$$0 = -\frac{d}{ds}\left(s^2X(s) - x'(0)\right) - \left(3\frac{d}{ds} + 1\right)\left(sX(s)\right) + 3X(s)$$

$$= -\left(2sX(s) + s^2X'(s)\right) - \left[3\left(X(s) + sX'(s)\right) + sX(s)\right] + 3X(s)$$

$$= -3sX(s) - (s^2 + 3s)X'(s)$$

So $\frac{X'(s)}{X(s)} = \frac{-3s}{s^2+3s}$ $\left(\text{or } \frac{dX/ds}{X} = \frac{-3s}{s^2+3s} \Rightarrow \frac{dX}{X} = \frac{-3s}{s^2+3s}ds\right)$

So $\int \frac{dX}{X} = \int \frac{-3s}{s^2+3s}ds = \int \frac{-3}{s+3}ds$

$$\ln X = -3\ln(s+3) + C$$

So $X(s) = C(s+3)^{-3} = \frac{C}{(s+3)^3}$ and hence $x(t) = \mathcal{L}^{-1}\{X(s)\} = Ct^2e^{-3t}$ for any $C \neq 0$

since $x(0) = 0$ regardless.

Section 7.5 Use the fact that $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$ to find

$$\mathcal{L}\{f(t)\} \quad \text{where } f(t) = \begin{cases} \cos \pi t, & \text{if } 0 \le t \le 2 \\ 0, & \text{if } t > 2. \end{cases}$$

$f(t) = f(t)\cdot(1-u(t-2))$, where $u(a) = \begin{cases} 1, & t > a \\ 0, & 0 \le t \le a. \end{cases}$

$= f(t) - u(t-2)f(t).$

But $f(t-2) = \cos\pi(t-2) = \cos\pi t = f(t)$.

So $f(t) = f(t) - u(t-2)f(t-2)$.

Therefore $\mathcal{L}\{f(t)\} = \frac{s}{s^2+\pi^2} - \frac{e^{-2s}\cdot s}{s^2+\pi^2} = \frac{s(1-e^{-2s})}{s^2+\pi^2}.$

Section 7.5 Consider the differential equation

$$y^{(4)} + 2y'' + y = 4t^3e^t; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

(a) Solve for the transform $Y(s) = \mathcal{L}\{y(t)\}$.
(Hint: You may need the formula $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$ or any other method.)

(b) Find the general form of the partial fraction decomposition of $Y(s)$. *You do not need to solve for the coefficients.*

(a) $\mathcal{L}\{y^{(4)} + 2y'' + y\} = \mathcal{L}\{4t^3e^t\}$

$s^4 Y(s) + 2s^2 Y(s) + Y(s) = \frac{24}{(s-1)^4}$

So $Y(s) = \frac{24}{(s-1)^4(s^2+1)^2}$

(b) $Y(s) = \frac{A}{(s-1)^4} + \frac{B}{(s-1)^3} + \frac{C}{(s-1)^2} + \frac{D}{(s-1)} + \frac{Es+F}{(s^2+1)^2} + \frac{Gs+H}{s^2+1}.$